Invariant-theoretic methods in scene analysis and structural mechanics

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Méthodes des invariants en analyse des scènes et en mécanique des structures

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Résumé. On décrit des applications possibles de la théorie des invariants aux problèmes non-résolus en géométrie appliquée. En particulier, on étudie les conditions projectives qui caractérisent les dessins plans des formes géométriques à trois dimensions, et les formes critiques des structures articulées.

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Abstract. We discuss possible applications of invariant theory to unsolved problems in applied geometry. In particular, we discuss projective conditions for correctness of plane drawings of 3-dimensional geometric forms, and for special mechanical behavior of bar-and-joint structures.

1. Introduction.

The purpose of this article is to describe two important, but as yet only partially developed, domains of application of invariant theory. It is hoped that by presenting a number of important classical examples and unsolved problems in an invariant-theoretic setting, we will encourage the active participation of experts in computer algebra and computational synthetic geometry to bring their talents to bear on the principal outstanding problems in two domains: the analysis of polyhedral scenes, and the mechanics of bar-and- joint structures.

First: scene analysis. Given a drawing (photograph, or other two-dimensional representation) of an object, we wish to evaluate its "natural dimension", that is, we wish to detect whether the drawing is the projection of a higher-dimensional object, and if so, of what (maximal) dimension, and with what relative heights of its component parts. The problem is one of "lifting" a lower-dimensional object

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into higher dimensions. Analogous problems occur when we try to "model" geometric forms, such as polyhedra, or more generally, polytopes. These problems are often made more difficult if models with additional properties are required (inscribed, or convex polyhedra, configurations with spatial symmetries, etc.).

Second: mechanics of structures and linkages. We wish to evaluate the rigidity or stability of architectural frameworks. One must be able to predict what unstable situations to avoid, and how to correct a design which has been found to be faulty or weak. For engineering applications, we need to predict the number of degrees of infinitesimal freedom of a mechanical linkage, and to predict those singular positions in which its number of degrees of freedom increases, or in which a particular finite motion becomes blocked.

By presenting a variety of examples and problems in these two areas of application, we shall try to sketch the present interface between practical geometric reasoning and computer algebra.

2. Some basic concepts in Invariant Theory

The points, lines, planes, ..., the *flats* (or *extensors*) of dimension 0, 1, 2, ..., in a real projective geometry P of dimension $n-1$, are in one-one correspondence with the linear subspaces (of dimensions 1, 2, 3, ..., respectively) in a real vector space V of dimension n. A flat of dimension $k-1$ is said to be of *rank* k. (We represent vectors in coordinate form, relative to a standard basis for the vector space.) Projective points are *coordinatized* (uniquely, up to a non-zero scalar multiple) by any non-zero vector in their associated 1-dimensional subspace.

Let N be the index set $N = \{1, 2, \dots, n\}$ of coordinate places, and for any value of k, $0 \leq k \leq n$, let $\binom{N}{k}$ be the set of ordered k-element subsets of N, the elements of these subsets being in *increasing* order. For any two *disjoint* subsets $A \in \binom{N}{j}, B \in \binom{N}{k}$, define

$$sign\ (A, B) = +1 \text{ or } -1,$$

according to the sign of the permutation which takes "A then B" (both sets separately in increasing order, with A before B) to the union "$A \cup B$" (merged in increasing order). We shall use "square" cups and caps to indicate disjoint union and the complementary notion, co-disjoint intersection:

$$A \sqcup B = C \iff A \cup B = C \text{ and } A \cap B = \emptyset,$$
$$A \sqcap B = C \iff A \cap B = C \text{ and } A \cup B = N.$$

Given any flat T of rank k in P, choose a basis for the corresponding subspace of V, and form a $k \times n$ matrix, using the basis vectors as rows. Then any subset $A \in \binom{N}{k}$ singles out k columns of this matrix, and we may compute the determinant of the corresponding $k \times k$ submatrix. This scalar value is called the *A-coordinate* of the flat T, and is written A_T. These coordinates,

$$T_A \text{ for } A \in \binom{N}{k},$$

are called *homogeneous, or Grassmann-Plücker, coordinates* of T, and are uniquely determined, up to a non-zero scalar multiple, by the subspace T, independent of the choice of basis for T. (Extending by anticommutativity, we can obtain coordinates indexed by *arbitrary* subsets of size k. For $k = 3$ for instance, $S_{378} = S_{837} = -S_{738}$.)

The basic operations of synthetic projective geometry are easily defined in terms of the Grassmann-Plücker coordinates. The *join* of a flat S of rank i with a flat T of rank j is the $(i + j)$-flat $S \vee T$ with coordinates

$$(S \vee T)_C = \sum_{A \sqcup B = C} sign\,(A, B)\; S_A T_B, \tag{1}$$

the sum being over all expressions of C as the disjoint union of subsets A and B of cardinality i and j, respectively. In terms of subspaces, the join of two flats is the linear span of their union. It is conventional to call this operation "wedge", and to employ the notation "$\wedge$", but we prefer the notation "$\vee$" as an aid to geometric intuition*. The join of a set of points, we abbreviate to a simple concatenation: $p \vee q \vee r$ becomes pqr. The join of any *dependent* set of points is equal to 0.

The *meet* of a flat S of rank i with a flat T of rank j is the $(i + j - n)$-flat $S \wedge T$ with coordinates

$$(S \wedge T)_C = \sum_{A \sqcap B = C} sign\,(A \backslash C, B \backslash C)\; S_A T_B, \tag{2}$$

* It is not clear apriori that the join, meet, and complement of flats are flats. This must be proven. See below, how "p-relations" distinguish the coordinates of extensors from those of antisymmetric tensors, in general.

the sum being over all expressions of C as the co-disjoint intersection of subsets A and B of cardinality i and j, respectively. (The symbol $\backslash$ denotes set-theoretic difference.)

The *complement* of a flat S of rank k is the flat S^* of rank $n-k$ with coordinates

$$(S^*)_{N\backslash A} = \; sign\,(A, N\backslash A)\, S_A. \tag{3}$$

Two flats are complementary if and only if their associated subspaces are orthogonal complementary, relative to inner product formed using the standard basis. This operation "*" is usually called the "Hodge star complement".

For instance, in a space of rank 4, let p be a point, S a line, and Q a plane, with coordinates

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ a & b & c & d \end{pmatrix},$$

$$S = \begin{pmatrix} 12 & 13 & 14 & 23 & 24 & 34 \\ s & t & u & v & w & x \end{pmatrix},$$

$$Q = \begin{pmatrix} 123 & 124 & 134 & 234 \\ h & i & j & k \end{pmatrix},$$

The join $p \vee S$ of the point p with the line S is a plane, with coordinates (given as a column vector, for lack of space)

$$(p \vee S) = \begin{pmatrix} 123: & p_1 S_{23} - p_2 S_{13} + p_3 S_{12} \\ 124: & p_1 S_{24} - p_2 S_{14} + p_4 S_{12} \\ 134: & p_1 S_{34} - p_3 S_{14} + p_4 S_{13} \\ 234: & p_2 S_{34} - p_3 S_{24} + p_4 S_{23} \end{pmatrix}$$

The meet $S \wedge Q$ of the line S with the plane Q is a point, with coordinates

$$(S \wedge Q) = \begin{pmatrix} 1: & S_{12}Q_{134} - S_{13}Q_{124} + S_{14}Q_{123} \\ 2: & S_{12}Q_{234} - S_{23}Q_{124} + S_{24}Q_{123} \\ 3: & S_{13}Q_{234} - S_{23}Q_{134} + S_{34}Q_{123} \\ 4: & S_{14}Q_{234} - S_{24}Q_{134} + S_{34}Q_{124} \end{pmatrix}$$

And finally, S^*, the complement of the line S, is a line with coordinates

$$S^* = \begin{pmatrix} 12 & 13 & 14 & 23 & 24 & 34 \\ S_{34} & -S_{24} & S_{23} & S_{14} & -S_{13} & S_{12} \end{pmatrix}.$$

The most commonly used properties of these synthetic operations are analogues of the Boolean identities:

$$\begin{bmatrix} (S \vee T)^* = S^* \wedge T^* \\ (S \wedge T)^* = S^* \vee T^* \\ S^{**} = (-1)^{k(n-k)} S \end{bmatrix} \tag{4}$$

The *bracket* of n points $a, b, \ldots, e$ in a projective space of rank n, written $[ab \ldots e]$, is the determinant of the $n \times n$ matrix of their coordinates. (Clearly, the value of the bracket depends on the vector representations chosen for the individual points; in any expression involving brackets, we assume that the same representation has been used for all instances of any given point.) For flats given as joins of points, the bracket can be used to express their meet. For instance, if $S = abcd$, $T = qrs$ in a space of rank 5, then

$$S \wedge T = qr[abcds] - qs[abcdr] + rs[abcdq]$$

or equivalently

$$S \wedge T = [abqrs]cd - [acqrs]bd + [adqrs]bc + [bcqrs]ad - [bdqrs]ac + [cdqrs]ab$$

The signs of the additive terms are the signs $sign\ (qr, st), \ldots$ of the *splits* $S = I \sqcup J$ or $T = I \sqcup J$ of the ordered sets S or T of points into two parts, one of which is the right size to "fill" the bracket:

$$S \wedge T = \sum_{I \sqcup J = T} sign\ (I, J)\ \ I[SJ] = \sum_{I \sqcup J = S} sign\ (I, J)\ \ [IT]J \tag{5}$$

This identity was used in [Doubilet et al, 1974] to define meet in terms of bracket and join, as the main axiom of the *Cayley algebra* of projective flats.

The minors of any rectangular matrix obey certain polynomial equalities. Consider the $k \times n$ matrix whose rows give the coordinates of k points in a space of rank n. If $k \leq n$, the polynomial relations are called the *p-relations* among the Grassmann-Plücker coordinates of the space spanned by those k points. If $k \geq n$, the polynomial relations are called *generic first-order syzygies* among the points. For the discussion which follows, we will not need a complete introduction to p-relations and syzygies. It suffices to recognize that the relations among the six 2×2 minors of a 2×4 matrix give both the p-relation

$$S_{12}S_{34} - S_{13}S_{24} + S_{14}S_{23} = 0, \tag{6}$$

(obtained by analyzing the identity $S \vee S = 0$) and the generic first-order syzygy

$$[ab][cd] - [ac][bd] + [ad][bc] = 0, \tag{7}$$

for four collinear points. Deleting all appearances of the point d from this expression, we obtain

$$\delta_{abc}: \quad [ab]c - [ac]b + [bc]a = 0, \tag{8}$$

the unique linear relation (up to an overall scalar multiple) that holds between three collinear points. This vector expression, we also call a *generic syzygy.* The adjective "generic" is used because these syzygies hold for *all* positions of points in the projective spaces in question. An analogous expression holds, for instance, when three points a, b, c "happen" to become collinear in the plane. For any additional point d,

$$a[bcd][-b[acd] + c[abd] = 0.$$

This syzygy is not generic; it is the specialization of the generic syzygy

$$\delta_{abcd}: \quad a[bcd] - b[acd] + c[abd] - d[abc] = 0 \tag{9}$$

to the case where $[abc] = 0$.

The relations among the ten 3×3 minors of a 3×5 matrix give three independent p-relations which determine three coordinates of an extensor T of step 3 in a space of rank 5 in terms of the remaining seven coordinates. For instance, $T_{145}, T_{245}, T_{345}$ can be calculated from the rest, using the p-relations*

$$\begin{aligned}
Rel_{\,12/1345}: &\quad +T_{123}T_{145} - T_{124}T_{135} + T_{125}T_{134} = 0 \\
Rel_{\,12/2345}: &\quad -T_{123}T_{245} + T_{124}T_{235} - T_{125}T_{234} = 0 \\
Rel_{\,13/2345}: &\quad -T_{123}T_{345} + T_{134}T_{235} - T_{135}T_{234} = 0
\end{aligned}$$

* The notation $Rel_{\,12/1345}$ suggests the structure of the p-relation. The terms are obtained choosing one of the four indices 1345, successively and with alternating signs, to go after the pair 12.

For any five general points a, b, c, d, e in the plane, rank 3, we have independent syzygies arising from the linear dependencies δ_{abcd} and δ_{abce} among vectors coordinatizing four coplanar points:

$$\begin{aligned}\delta_{abcd}: \quad & a[bcd] - b[acd] + c[abd] - d[abc] = 0,\\ \delta_{abce}: \quad & a[bce] - b[ace] + c[abe] - e[abc] = 0.\end{aligned}$$

In the form of a polynomial identity, the first of these two generic syzygies takes the form

$$\delta_{abcd}: \quad [axy][bcd] - [bxy][acd] + [cxy][abd] - [dxy][abc] = 0,$$

where x and y can be arbitrary points, not necessarily distinct from a, b, c, d. Generic linear dependencies among first order syzygies are called *generic second order syzygies*. Continuing the above example of five points in a space of rank 3, we find that the syzygies δ_{abcd}, δ_{abce}, and δ_{abde} are related by the generic second order syzygy:

$$[abe]\delta_{abcd} - [abd]\delta_{abce} + [abc]\delta_{abde} = 0. \tag{10}$$

We can see this by looking at the matrix of coefficients of the three syzygies in question,

$$\begin{array}{c} \\ \delta_{abcd} \\ \delta_{abce} \\ \delta_{abde} \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} [bcd] & -[acd] & [abd] & -[abc] & 0 \\ [bce] & -[ace] & [abe] & 0 & -[abc] \\ [bde] & -[ade] & 0 & [abe] & -[abd] \end{pmatrix} \end{array} \begin{array}{c} \text{coef} \\ \begin{pmatrix} [+abe] \\ -[abd] \\ [+abc] \end{pmatrix} \end{array} \tag{11}$$

and verifying that the stated linear combination (shown in the last column) produce 0 in columns a and b, because the resulting expressions are generic first-order syzygies.

3. Scene Analysis

In the remarks which follow, we shall be dealing primarily with structures defined by incidence of flats (points, lines, planes) in three-dimensional space. We deal first with polyhedral scenes, then with more general configurations. Our attention is focused on projective invariants of such figures, and in particular on invariant properties of plane figures which permit their valid interpretation as projections of 3-dimensional figures. It is normally not difficult to obtain these invariant properties in the form of equations among polynomials in the homogeneous coordinates of points. What *is* difficult, however, is to arrive at synthetic geometric interpretations of such properties, to arrive at an intuitive and geometric "understanding" of the circumstances in which such invariant properties hold.

The key to geometric interpretation of invariant properties lies in the factorization of invariant polynomials as expressions in the more transparently "geometric" Cayley algebra, with "join" ($\vee$) and "meet" ($\wedge$) as operations on flats. By looking more closely at the process of analyzing polyhedral scenes, and at the companion problem of synthesizing 3-dimensional polyhedral forms with given planar projections, we will make clear why Cayley factorization of invariant polynomial forms is both a crucial and rather difficult problem.

An algorithm created by Tim McMillan and Neil White (see the article by White, in this collection) succeeds in factoring multilinear bracket polynomials as join-meet ($\vee$, $\wedge$) expressions in the Cayley algebra, whenever this is possible, and otherwise shows that no such expression exists. The situation is quite different for bracket polynomials which are not multilinear. The notion of anti-commuting variables (used by McMillan and White to identify *atomic sets*), no longer makes sense unless we are willing to try pairing, successively, each appearance of one variable with each appearance of another, and this independently for each additive term of the given polynomial! With such an approach, an already difficult algorithm for the multilinear case will assume nightmare proportions for general polynomials. It is perhaps better to look first for a alternative approach to the factorization problem, one which will hopefully respect the advice of geometric intuition.

3.1. EXAMPLE: PROJECTIVE CONDITIONS FOR A POLYHEDRAL SCENE

Figure 1.1 is a plane projection of a polyhedron with 7 points, $a, b, \ldots, g$, 11 edges $ab, \ldots, fg$, and 6 faces

$$A = abcd, B = cdef, C = bceg, D = adfg, E = abg, F = efg.$$

That the drawing is the correct projection of a spatial polyhedron is clear from the construction in Figure 1.2. As a start, any drawing of a tetrahedron ($cdxy$ in our figure) is correct. This tetrahedron $cdxy$ has been twice truncated through a

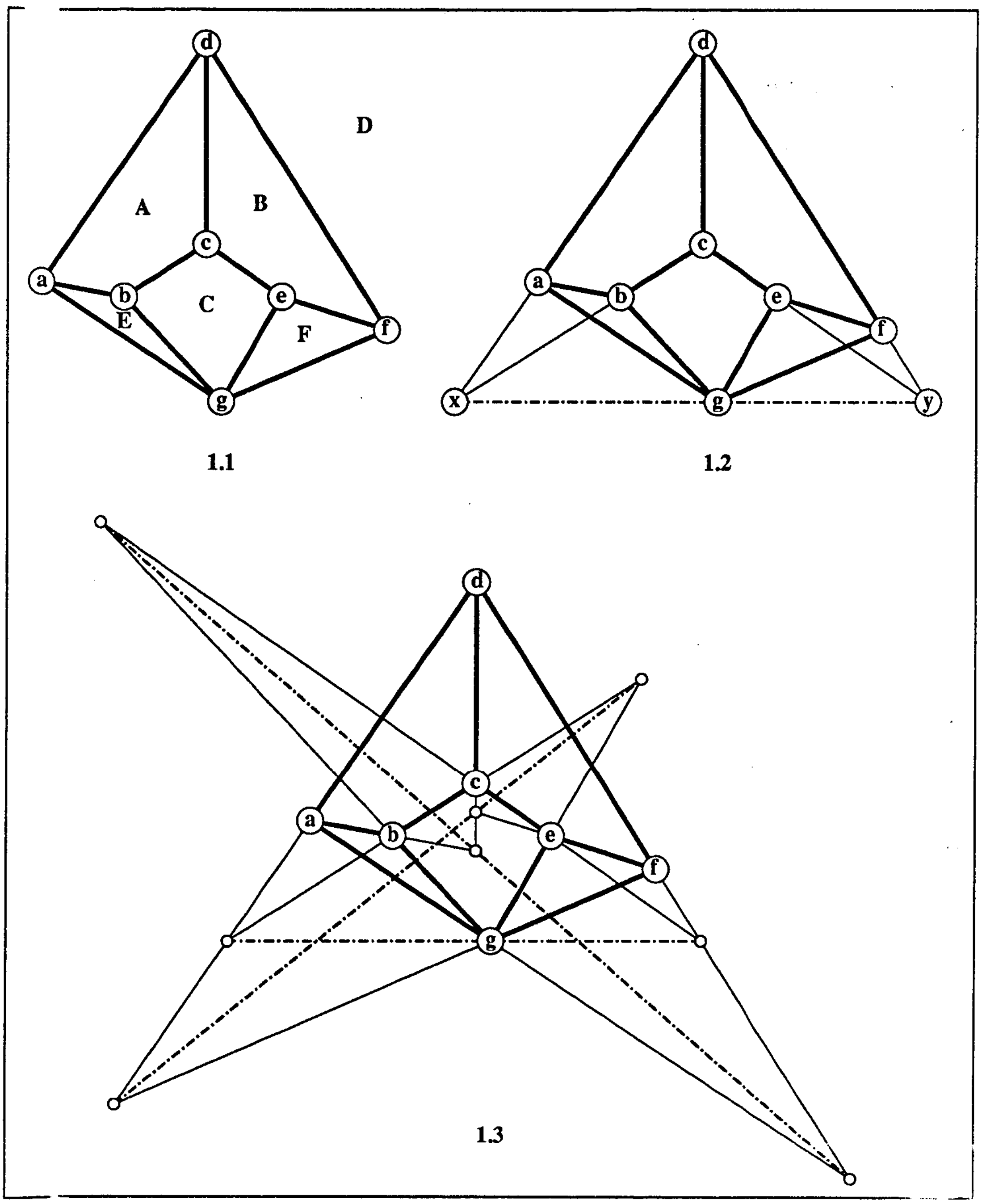

Figure 1. Projective conditions for a polyhedral projection.

point g on the edge xy. In a general drawing of those 7 points and 11 edges, the constructed points

$$x = ad \wedge bc \text{ and } y = ce \wedge df$$

are not necessarily collinear with the point g. A *necessary* condition for a correct drawing is given by the Cayley algebra expression

$$(ad \wedge bc) \vee g \vee (ce \wedge df) = 0. \tag{12}$$

Expression (12) affirms that the three points $ad \wedge bc$, g, $ce \wedge df$, each of which is necessarily on the line $C \wedge D$ in space, are collinear in projection. The expression is also *sufficient*, because it guarantees that the triangles abg and efg, drawn from the point g on the line xy, will correctly represent truncations of the tetrahedron $cdxy$.

Expanding formula (12) according to the procedure shown in equation (5) we obtain

$$\begin{aligned}&(b[adc] - c[adb]) \vee g \vee ([cdf]e - [edf]c)\\ &= [bge][adc][cdf] - [bgc][adc][edf] - [cge][adb][cdf]\end{aligned} \tag{13}$$

Expression (13) is a homogeneous polynomial of brackets of points, thus a projectively invariant expression. It is linear in variables a, b, e, f, g, quadratic in c, d.

Expression (12) is not the only such expression of the fact that the given drawing is a correct polyhedral projection. Consider also the process of constructing the projected line of intersection of planes B and E. These two planes have points $ab \wedge cd$, $bg \wedge ce$, and $ag \wedge df$ in common, as shown in Figure 1.3, so

$$(ab \wedge cd) \vee (bg \wedge ce) \vee (ag \wedge df) = 0 \tag{14}$$

will guarantee a correct polyhedral drawing. It expands to the bracket polynomial

$$\begin{aligned}&([acd]b - [bcd]a) \vee ([bce]g - [gce]b) \vee ([adf]g - [gdf]a)\\ &= -[bga][acd][bce][gdf] + [abg][bcd][gce][adf]\end{aligned} \tag{15}$$

This projectively invariant polynomial is of higher degree than that of expression (12). It is linear in e, f, quadratic in a, b, c, d, g. It can be reduced to the same degree by removing the factor of $[abg] = [bga]$ which appears (happily!) in expansion (15). But we must still inquire as to the geometric significance of this bracket factor, and decide whether it belongs in the "projective condition" for this polyhedron.

But first, consider one further expression of this condition, obtained by constructing the projected line of intersection of planes A and F. These planes have points $cd \wedge ef$, $bc \wedge eg$, and $ad \wedge fg$ in common, so, again in Figure 1.3,

$$(cd \wedge ef) \vee (bc \wedge eg) \vee (ad \wedge fg) = 0 \tag{16}$$

expresses the required condition. It expands to the bracket polynomial

$$\begin{aligned}(e[cdf] - f[cde]) \vee (e[bcg] - g[bce]) \vee (f[adg] - g[adf]) \\ = -[egf][cdf][bce][adg] + [feg][cde][bcg][adf]\end{aligned} \tag{17}$$

This polynomial is linear in a, b, quadratic in c, d, e, f, g. It can likewise be reduced to the degree of expression (12) by removal of the bracket factor $[egf]$ which appears as a factor of expression (17).

The three reduced polynomials

$$\begin{gathered}[acd][beg][cdf] + [acd][bcg][def] - [abd][cdf][ceg] \\ -[acd][bce][dfg] + [adf][bcd][ceg] \\ [adg][bce][cdf] - [adf][bcg][cde]\end{gathered} \tag{18}$$

all straighten to the same form,

$$[acd][bdf][ceg] - [acd][bce][dfg] - [abd][cdf][ceg], \tag{19}$$

so they express the same projectively invariant property. Call this invariant "Q", in whatever form it is written. Equations (12), (14), and (16) are thus

$$Q = 0, \quad [abg]Q = 0, \quad -[efg]Q = 0.$$

By dictate of symmetry, there are two reasonable choices for the "projective condition" for the given polyhedron, either $Q = 0$ or $-[abg][efg]Q = 0$. In fact, these conditions are the solutions to two different problems, both of which deserve attention. For want of a better term, we shall say these are the problems of finding realizations of *type I* and of *type II*, respectively.

We specify a spatial realization of type I by giving the height $h(a)$ of each vertex $a' = [a, h(a)]$ over its image a in the plane drawing, and we insist that sets of cofacial vertices be coplanar in space. There is a true spatial realization of the polyhedron (with its vertices not all lying on a single plane) that projects to the given drawing if and only if $Q = 0$, that is, if and only if condition (12) holds. Since polynomial (13) is projectively equivalent to factors of both polynomials (15) and (17), condition (12) *implies* conditions (14) and (16). In any drawing for which $Q = 0$, all three geometric constructions (of the projected intersections of planes C and D, B and E, A and F) will succeed.

If, in the given drawing, the face $E = abg$ ($F = efg$, resp.) is formed from three *collinear* points, then the Cayley algebra expression (14) (resp., (16)) will be zero even when there is no true spatial realization of the drawing. Caution! These "extra" bracket factors are not "visible" in the Cayley algebra expressions, nor "present" in the associated geometric constructions. But they must be removed from expressions like (15) and (17) if one wishes the condition for existence of non-trivial solutions of type I. It is worth noting also that if we start from the geometric ideas used to produce formulas (14) and (16), we do *not* obtain a Cayley factorization of Q. Indeed, the extensor ab, formed from points which occur linearly in Q, is not "atomic" in the sense of McMillan and White: the polynomial $Q + Q'$ does not straighten to zero, where Q' is obtained from Q by interchanging a and b.

We specify a spatial realization of type II by giving a height for each vertex, as before, and also a plane P_A for each face A; we insist that incidence between vertices a and faces A be preserved as incidence between points $a' = [a, h(a)]$ and planes $P(A)$. The vanishing of the polynomial $-[abg][efg]Q$ implies that either $Q = 0$, permitting a proper polyhedron with vertices not all on one plane, or $[abg][efg] = 0$, in which case at least one of the two triangular faces is formed from collinear points, and a degenerate spatial realization becomes possible. All the vertices lie on one plane P_1, the vertices of the triangle lie on a line L in P_1, and the triangular face is assigned to a plane $P_2 \neq P_1$ passing through the line L. Such a solution qualifies as a "true" spatial realization in this new context, because not all of the planes P_A coincide.

Look more closely at solutions of type I, in terms of syzygies relating cofacial points in a polyhedral drawing. Consider the 3×7 matrix $\mathcal{M}$ whose columns are the coordinate representations of the 7 points $a, \ldots, f$. Let $\mathcal{N}$ be the 4×7 matrix whose rows give the coefficients of the generic first order syzygies among the points on the faces $A = abcd$, $B = cdef$, $C = bceg$ and $D = adfg$, the faces with at least four incident points:

$$\begin{array}{c} \\ \delta_{\mathbf{abcd}} \\ \delta_{\mathbf{cdef}} \\ \delta_{\mathbf{bceg}} \\ \delta_{\mathbf{adfg}} \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{e} & \mathbf{f} & \mathbf{g} \end{array} \\ \begin{pmatrix} +[bcd] & -[acd] & +[abd] & -[abc] & 0 & 0 & 0 \\ 0 & 0 & +[def] & -[cef] & +[cdf] & -[cde] & 0 \\ 0 & +[ceg] & -[beg] & 0 & +[bcg] & 0 & -[bce] \\ +[dfg] & 0 & 0 & -[afg] & 0 & +[adg] & -[adf] \end{pmatrix} \end{array} \qquad (20)$$

If the points are given in standard projective coordinates, with third coordinate equal to 1, the rows of matrix $\mathcal{M}$ list the values of the coordinate axis projection functions x and y, and of the constant function 1, at the points $a, \ldots, g$. These rows thus generate the space of *linear functions* on the set $P = \{a, \ldots, g\}$. Each row of the matrix $\mathcal{N}$ is orthogonal to every linear function on P, and to every vector listing the heights to the vertices of a spatial realization in which the four-point sets on faces A, B, C, D are coplanar. So the rank of the space of spatial realizations of a given drawing of this polyhedron is equal to 7, the number of points, less the rank of the matrix N. For points $a, \ldots, g$ in general position (not satisfying the condition $Q = 0$) the matrix $\mathcal{N}$ has rank 4, and the only spatial realizations are trivial, with all points coplanar. It is only when the matrix $\mathcal{N}$ has rank three or less, that there can be any true spatial polyhedra which project to the given drawing.

For points $a, \ldots, g$ in general position, the row spaces $Row_{\mathcal{M}}$ and $Row_{\mathcal{N}}$ are orthogonal complementary subspaces of R^7. The Grassmann-Plücker coordinates of $Row_{\mathcal{M}}$ are simply brackets of triples of points. The coordinates of $Row_{\mathcal{N}}$ are Hodge-star complementary to those of $Row_{\mathcal{M}}$, so, by equation (3), and including the possible overall scalar factor λ (which may depend on the positions of $a, \ldots, g$),

$$[\ Row_{\mathcal{N}}]_{X \backslash A} = \lambda \ \ sign\ (A, X \backslash A)\ [A].$$

The only way for all these coordinates to vanish is for the entire configuration to be of rank < 3 (that is, $[A] = 0$ for *all* triples A of points), or for the scalar λ to be zero. We can compute the scalar factor λ by looking at any *one* minor of the matrix $\mathcal{N}$. Indeed, if we choose the set $X \backslash A$ of columns with care, we can obtain $[A]$ as an obvious factor of the resulting minor, and the scalar λ will be obtained without further effort. Here, for instance, we can see that $[bcd]$ will be a factor of the $aefg$ minor of the matrix $\mathcal{N}$. We obtain

$$\begin{aligned} \lambda\,[bcd] = \lambda\,\mathcal{M}_{bcd} = \; & sign\,(aefg, bcd)\;\; \mathcal{N}_{aefg} \\ = [bcd]\,&([cdf][bce][adg] - [cde][bcg][adf]). \end{aligned} \tag{21}$$

By this equation, λ is equal to one of the three forms of the condition Q we obtained above (see equation 18). The condition $Q = 0$ is precisely the condition for the syzygies on the faces of the polyhedral drawing to be dependent (rank 3).

This approach, using syzygies on the faces, is suggestive of general methods of attack for problems in scene analysis. The subject has been more fully developed in papers on geometric homology [Crapo and Ryan, 1986; Crapo, 1989].

In what follows, when we speak of *projective conditions*, we mean conditions for spatial realizations of type I. Also, before leaving this first example, we should point out that the polyhedron in question was selected to illustrate what happens when the heights of p points are controlled by $p - 3$ possibly independent coplanarity constraints, thus making *one* projective condition necessary for a true spatial realization. This situation occurs when

$$f_4 + 2f_5 + \ldots = p - 3,$$

(f_i being the number of faces with i points). Adding $3f_3 + 3f_4 + \ldots = 3f$, where f is the total number of faces, we find

$$2e = 3f_3 + 4f_4 + 5f_5 + \ldots = 3f + p - 3,$$

where e is the number of edges. Subtracting a multiple $2(e = f + p - 2)$ of the Euler-Poincaré relation, we have $p - f = 1$. In general, the difference $p - f$ gives the expected number of projective conditions for spatial realizations of type I, for polyhedra.

3.2. CALOTTE CONDITIONS

A second example, also drawn from scene analysis, is the projective condition that a cycle of lines radiating from the vertices of a plane n-gon be the correct projection of a ring of faces surrounding an n-gonal piece of plane in space, the spatial figure being not entirely coplanar. We call such a figure a calotte, as shown in Figure 2.1.

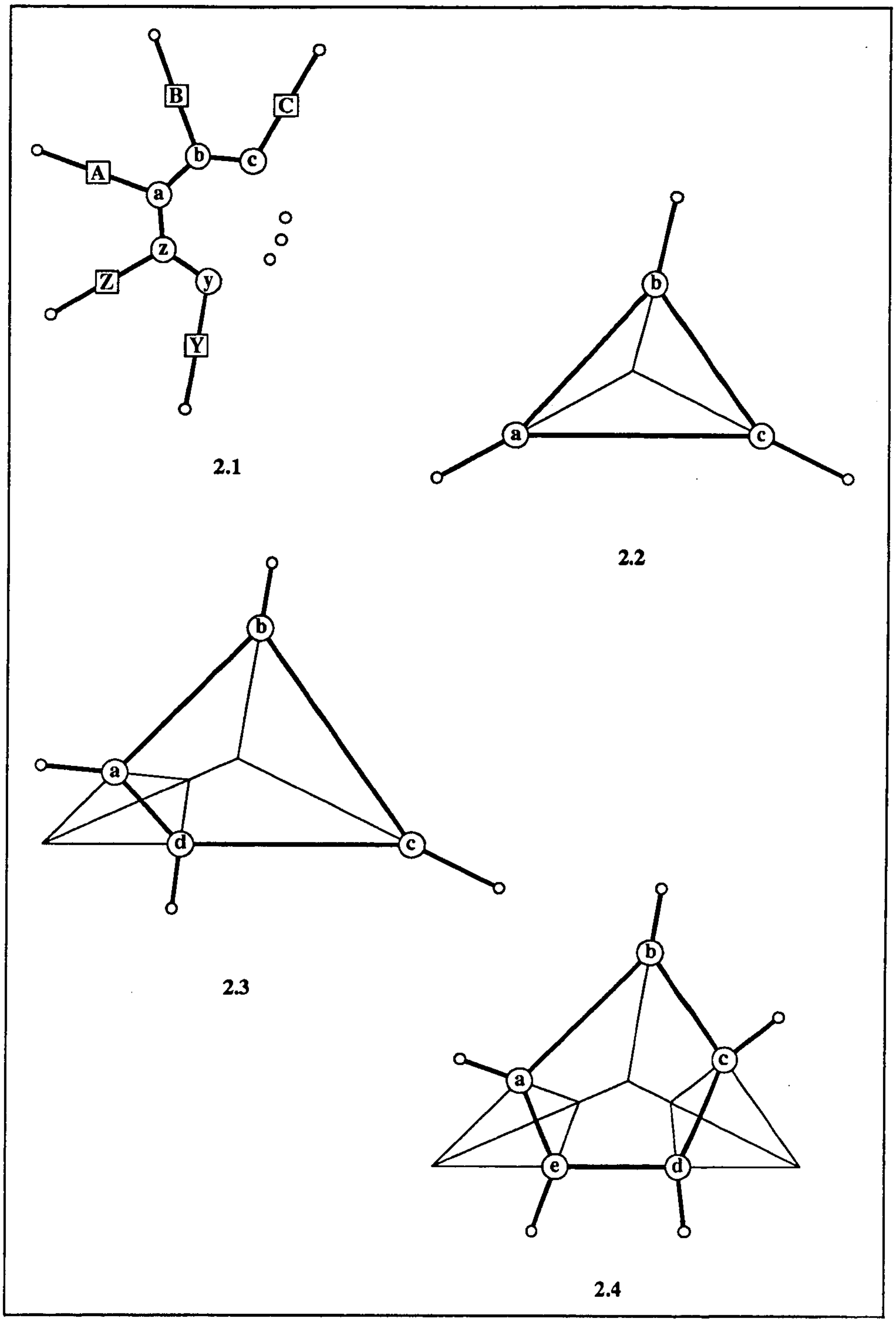

Figure 2. "Roof" constructions for calottes.

Theorem 1. *The projective condition for the n-calotte, $n \geq 4$, is*

$$[aB][bC][cD]\ldots[zA] \ + \ (-1)^{n-1}\,[Ab][Bc]\ldots[Yz][Za] = 0 \tag{22}$$

For $n = 3: \ A \wedge B \wedge C = 0.$

Proof: Let $a', \ldots, z'$ be general points on the lines $A, \ldots, Z$, respectively. Let $\mathcal{N}$ be the matrix whose rows list the coefficients of the generic first order syzygies on the exterior faces $AB, BC, \ldots, ZA$ of the calotte:

$$\begin{array}{c} \\ \delta_{\mathbf{AB}} \\ \delta_{\mathbf{BC}} \\ \vdots \\ \delta_{\mathbf{YZ}} \\ \delta_{\mathbf{ZA}} \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathbf{a'} & \mathbf{a} & \mathbf{b'} & \mathbf{b} & \ldots & \mathbf{z'} & \mathbf{z} \end{array} \\ \begin{pmatrix} [aB] & -[a'B] & [Ab] & -[Ab'] & \ldots & 0 & 0 \\ 0 & 0 & [bC] & -[b'C] & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & [Yz] & -[Yz'] \\ [Za] & -[Za'] & 0 & 0 & \ldots & [zA] & -[z'A] \end{pmatrix} \end{array} \tag{22}$$

There is one generic syzygy on each of the outer faces $AB \ldots ZA$. The syzygies on the central n-gon form a space of rank $n-3$. Since the n syzygies on the outer faces are independent, and since there are $2n$ points, there can only be a true spatial calotte over the given projection if the total rank of these syzygies is $\leq 2n-4$. If this is the case, then either the syzygies on the outer faces are dependent (rank $\leq n$), or some linear combination of those syzygies is a syzygy among the n points of the central polygon.

Up to a scalar multiple, the only linear combination of rows $\delta_{AB} \ \ldots \ \delta_{ZA}$ which is zero in columns $b', c', \ldots, z'$ is

$$\begin{array}{c} + \ [bC][cD][dE]\ldots[zA]\ \delta_{AB} \\ - \ [Ab][cD][dE]\ldots[zA]\ \delta_{BC} \\ + \ldots \\ + \ (-1)^n\ [Ab][Bc][Cd]\ldots[Xy][zA]\ \delta_{YZ} \\ + \ (-1)^{n-1}\ [Ab][Bc][Cd]\ldots[Yz]\ \delta_{ZA} \end{array} \tag{23}$$

This particular linear combination also produces a 0 in column a', and thus a syzygy among the inner points $a, \ldots, z$, if and only if

$$[aB][bC][cD]\ldots[zA] \ + \ (-1)^{n-1}\,[Ab][Bc]\ldots[Yz][Za] = 0 \tag{24}$$

For the syzygies on the outer faces to be dependent, this same linear combination must also produce a 0 in rows $b, \ldots, z$. That is:

$$\begin{gathered}-[Ab'][bC]+[Ab][b'C]=0 \ ,\ldots,\ -[Yz'][zA]+[Yz][z'A]=0, \\ \text{or equivalently,} \quad A \wedge B \wedge C=0 \ ,\ldots,\ X \wedge Y \wedge A=0.\end{gathered} \tag{25}$$

Equations (24) and (25) together imply that the linear combination (23) produces a 0 in column a. It follows that the calotte has a spatial realization of type I if and only if either all the lines $A, \ldots, Z$ are concurrent, radial from a single central point, or else condition (24) holds.

For $n = 3$ (see Figure 2.2), the polyhomial of condition (22) is

$$[Ab][Bc][Ca]-[aB][bC][cA]$$

which (relative to the order $a'ab'bc'c$) straightens to

$$-[a'ab'][abc][bc'c]+[a'ab][abc][b'c'c],$$

revealing a factor of $[abc]$ and a factorization

$$[abc](-[Ab'][bC]+[Ab][b'C])=[abc](A \wedge B \wedge C).$$

Vanishing of the bracket $[abc]$, with $A \wedge B \wedge C \neq 0$, does not produce a true spatial realization of type I, so the condition for a 3-calotte is simply $A \wedge B \wedge C = 0$. □

Do these conditions, for $n \geq 4$, have a Cayley factorization? The 4-calotte in Figure 2.3 has as projective condition

$$[Ab][Bc][Cd][Da]+[aB][bC][cD][dA]=0. \tag{26}$$

Assume the given drawing is the correct projection of a non-trivial polyhedral scene. Then it will be possible to construct the intersection of faces $AB = a'ab'b$ and $CD = c'cd'd$. The resulting line of intersection must contain the three points $A \wedge D$, $ab \wedge cd$, $B \wedge C$, so a necessary and sufficient condition for a correct drawing is that these three points be collinear:

$$\begin{aligned} 0 &= (A \wedge D) \vee (ab \wedge cd) \vee (B \wedge C) \\ &= (d'[Ad]-d[Ad']) \vee (c[abd]-d[abc]) \vee (c'[Bc]-c[Bc']) \\ &= -[d'C][Ad][abd][Bc]-[Dc'][Ad][abc][Bc] \\ &\quad +[Dc][Ad][abc][Bc']+[dC][Ad'][abd][Bc] \end{aligned}$$

which has exactly the same straightened form as equation (24), a sum of ten products of brackets. As Figure 2.3 shows, collinearity of the points $A \wedge D$, $ab \wedge cd$, $B \wedge C$ permits us to construct a *roof* of plane polygons over the central polygonal region.

The n-calotte condition has degree 2 in each of the points on the perimeter of the n-gon, degree 1 in the variable points taken to generate the radial lines. But an analogous roof construction requires a higher degree in several of the variables. We conjecture that *the calotte condition is not Cayley factorable for $n \geq 5$, but becomes so when multiplied by a product of $n - 4$ brackets.* Consider the case $n = 5$. The calotte condition for the calotte in Figure 2.4 is

$$0 = [Ab][Bc][Cd][De][Ea] - [aB][bC][cD][dE][eA] \tag{27}$$

which straightens to a sum of eighteen products of brackets. A corresponding roof construction can be written

$$0 = ((A \wedge E) \vee (ab \wedge de)) \wedge ((C \wedge D) \vee (bc \wedge de)) \wedge (B)$$

which expands to an expression in some ten additive terms, and straightens to the same form as formula (27), once formula (27) has been multiplied by the bracket $[bde]$, to bring it up to the proper degree in each variable point. For further basic material concerning such conditions, see [Baracs 1973 –].

3.3. EXAMPLE: THE 3 BY 3 GRID

Say we wish to solve the problem of drawing a correct planar projection of a grid formed by three lines A, B, C meeting three lines D, E, F, the entire figure being *skew* (non-coplanar) in 3-space. We know that such a figure is possible in space, because from *any* point d on the line A we may draw a unique line D meeting the skew lines B, C, as follows. Since the desired line D passes through d and meets B, it lies in the plane $d \vee B$. Since it meets C, it must also lie in the plane $d \vee C$. Thus D is uniquely determined as the line of intersection $D = (d \vee B) \wedge (d \vee C)$, provided that B and C are not coplanar, or at least not coplanar with d. What algorithmic approach can be taken to the problem of drawing a plane projection of such a grid?

One procedure is to make the construction directly in 3-space. Given choices for the projected positions a, b, c, d, e, f, g in the plane, with sets abc, adg, def each collinear, our problem is to determine the position of the points h, i. We do so by constructing points

$$a' = (a_1, a_2, s(a), 1), \ \dots \ , i' = (i_1, i_2, s(i), 1)$$

which project to the given positions of $a, \dots, g$, and which reveal the correct positions of the points h, i. First we choose random heights $s(a), s(b), s(d), s(e)$, making sure only that the four points a', b', d', e' are *not* coplanar in space. That is, we make sure that

$$s(a)[bde] - s(b)[ade] + s(d)[abe] - s(e)[abd] \neq 0.$$

The heights of c', f', g' can then be calculated, using the fact that abc, adg, def are collinear in space. For example, to determine $s(c)$, we solve for $s(c)$ in the equation

$$s(a)[bcd] - s(b)[acd] - s(c)[abd] = 0.$$

The point i' is obtained from the construction

$$i' = (g' \vee b' \vee e') \wedge (c' \vee f'),$$

the line C is equal to $g' \vee i'$, and the final point h can be obtained as

$$h' = (g' \vee i' \vee f') \wedge (b' \vee e').$$

Dropping the third coordinates of h' and of i', we obtain the required projections h and i. What is perhaps surprising is that the position of the projected line C is independent of the choices of heights $s(a), s(b), s(d), s(e)$ used in this construction. Subsequent constructions will show this to be the case.

A second construction, although motivated by spatial reasoning, takes place entirely in the plane. We reason that the set of nine points $a \dots i$ can be resolved into three planes (in several ways). For instance, the lines $a'b'c'$ and $b'e'h'$ are coplanar in a skew spatial model which projects to any given such drawing, as are $a'd'g'$ and $d'e'f'$, and as are $c'f'i'$ and $g'h'i'$. Call these planes $A = a'b'c'e'h'$, $B = a'd'e'f'g'$, and $C = c'f'g'h'i'$. These three sets are separately coplanar on non-coincident planes, if and only if the six sets $abc, def, \dots cfi$ are separately straight, not all coplanar. The reason: three collinear points (say abc)

have collinear preimages $a'b'c'$ with respect to vertical projection if and only if $a'b'c'$ are coplanar on a non-vertical plane. The line of intersection $A \wedge B$ passes through the points a' and e', so its projection must lie along the line $a \vee e$ in the plane. Similarly, the projection of $A \wedge C$ is along the line $c \vee h$, and that of $B \wedge C$ is along the line $f \vee g$. If the planes A, B, C are not equal, then they will intersect in a point that lies on the three lines $A \wedge B$, $A \wedge C$, $B \wedge C$. The projection of this point will then lie on all three lines $a \vee e$, $c \vee h$, $f \vee g$. See Figure 3. So the construction in the plane can be carried out as follows. Choose arbitrary (but non-collinear) positions for points a, b, d, e. Choose arbitrary (but distinct) positions for c on the line $a \vee b$, for f on the line $d \vee e$, and for g on the line $a \vee d$. Construct the point $x = (f \vee g) \wedge (a \vee e)$. Locate h at the point $(c \vee x) \wedge (b \vee e)$, the line C at $g \vee h$, and i at the point $(g \vee h) \wedge (c \vee f)$.

A third construction proceeds in terms of syzygies. Again, we ask the equivalent question: "Can the three sets $A = a'b'c'e'h'$, $B = a'd'e'f'g'$, $C = c'f'g'h'i'$ be separately coplanar, jointly skew, over projected point positions $a, \ldots, i$?" There are three first order syzygies of minimal support on each of these five-point sets. For instance, on A we have the following syzygies, where x and y are arbitrary points in the plane.

$$\begin{array}{c} \\ \delta_{\mathbf{abc}} \\ \delta_{\mathbf{beh}} \\ \delta_{\mathbf{aceh}} \end{array} \begin{array}{c} \begin{array}{ccccc} \quad\mathbf{a}\quad & \quad\mathbf{b}\quad & \quad\mathbf{c}\quad & \quad\mathbf{e}\quad & \quad\mathbf{h}\quad \end{array} \\ \begin{pmatrix} +[bcx] & -[acx] & +[abx] & & \\ & +[ehy] & & -[bhy] & +[bey] \\ +[ceh] & & -[aeh] & +[ach] & -[ace] \end{pmatrix} \end{array}$$

These three syzygies have rank 2. If the space of syzygies supported on the sets A, B, C have the maximum possible rank $2 \times 3 = 6$, then, as conditions on the heights $s(a), \ldots s(i)$ they will permit only $9 - 6 = 3$ independent solutions. These will form the 3-dimensional space of globally linear solutions, where the points $a', \ldots, i'$ are all coplanar. For there to be a proper (skew) spatial model, the dimension of the space of syzygies supported on the sets A, B, C must be no higher than 5, and some dependence must be found between syzygies supported on those three sets. If a syzygy supported on C is a linear combination of syzygies supported on A and on B, it cannot involve the point i, which is not in $A \cup B$. There is only one such syzygy, up to an overall scalar multiple, namely the dependence δ_{cfgh}. The same reasoning reveals that the syzygies supported on A and B, and

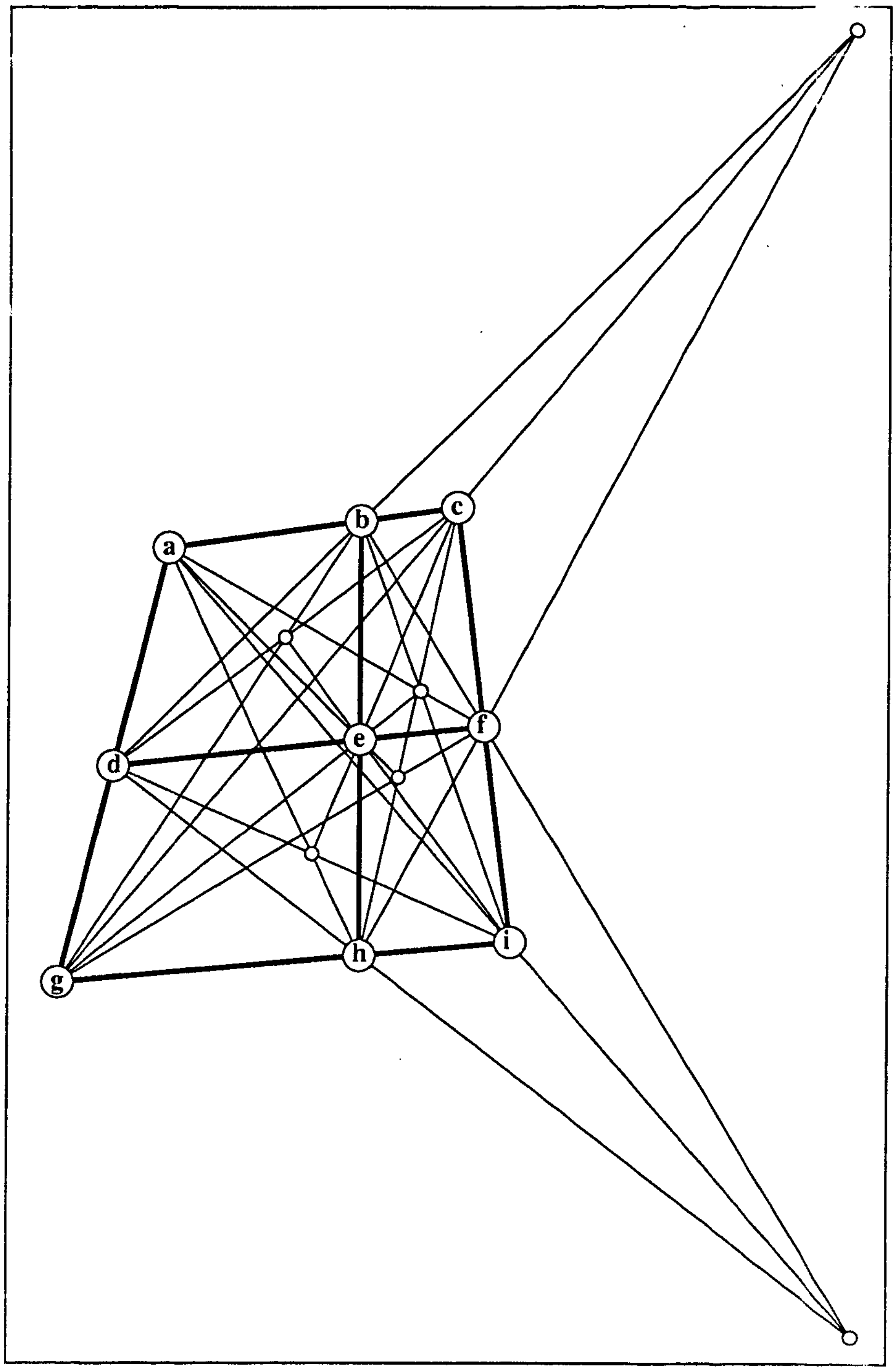

Figure 3. Six equivalent projective conditions for a skew grid.

involved in this relation, must be multiples of δ_{aceh}, δ_{aefg}, respectively. That is, the three syzygies δ_{aceh}, δ_{aefg}, δ_{cfgh} must be dependent. Using columns a and c, we determine the scalars of such a linear relation:

	a	**c**	**e**	**f**	**g**	**h**	coef
$\delta_{\mathbf{aceh}}$	$+[ceh]$	$-[aeh]$	$+[ach]$			$-[ace]$	$+[efg][fgh]$
$\delta_{\mathbf{aefg}}$	$+[efg]$		$-[afg]$	$+[aeg]$	$-[aef]$		$-[ceh][fgh]$
$\delta_{\mathbf{cfgh}}$		$+[fgh]$		$-[cgh]$	$+[cfh]$	$-[cfg]$	$+[efg][aeh]$

We then verify that these scalars give a linear dependence, by checking the other columns. For instance, in column f we find

$$-[aeg][ceh][fgh] - [cgh][efg][aeh].$$

Using the generic syzygy $(eg|afgh)$: $[aeg][fgh] = -[efg][agh] + [afg][egh]$, we may rewrite this expression as

$$+[efg][agh][ceh] - [afg][egh][ceh] - [cgh][efg][aeh],$$

and using the generic syzygy $(gh|aceh)$: $[agh][ceh] + [gch][aeh] + [ach][egh] = 0$, we reduce it to the form

$$\begin{aligned}&-[efg][ach][egh] - [afg][egh][ceh]\\ &= [egh]([fga][ech] - [fge][ach])\\ &= [egh](fg \wedge ea \wedge ch)\end{aligned}$$

which we know to be zero whenever the stated condition holds. Similar calculations can be carried out in columns g and h, to verify the second order syzygy:

$$[efg][fgh]\delta_{aceh} - [ceh][fgh]\delta_{aefg} + [efg][aeh]\delta_{cfgh} = 0.$$

As we have observed before (when discussing polyhedral projections) it is *not* necessary to verify all the columns in this tedious fashion. It suffices to notice that the three first order syzygies δ_{aceh}, δ_{aefg}, δ_{cfgh}, restricted to columns a, c, e, f, g, h, form a matrix $\mathcal{M}$ whose rows are orthogonal to all the rows of the 3×6 matrix $\mathcal{N}$ whose columns coordinatize the points a, c, e, f, g, h. If the rows of $\mathcal{M}$ are independent, then the row spaces of $\mathcal{M}$ and $\mathcal{N}$ are orthogonal complementary in R^6, and have Grassmann coordinates which are Hodge-star complementary. That is, for any 3-element subset A of $X = acefgh$, the A-coordinate of the row space of $\mathcal{N}$ is merely $sign\ (A, X\backslash A)\ \ [X\backslash A]$, up to an overall scalar multiple λ, independent of the choice of A. We can evaluate the scalar λ for any one subset A, say at $A = ace$. There, we find the Grassmann coordinate

$$N_{ace} = \det \begin{vmatrix} +[ceh] & -[aeh] & +[ach] \\ +[efg] & 0 & -[afg] \\ 0 & +[fgh] & 0 \end{vmatrix} = [fgh]\left([ceh][afg] + [ach][efg]\right).$$

Since $fgh = X\backslash ace$, and $sign\ (ace, fgh) = +1$, we have

$$\lambda = [cha][efg] - [che][afg] = ch \wedge ea \wedge fg.$$

The three syzygies are dependent if and only if their join as 6-vectors (or as 9-vectors relative to the set $\{a, \dots i\}$ in the Grassmann algebra is zero, if and only if the scalar λ is zero, ie., if and only if the three lines ch, ae, fg are concurrent.

Observe that the choice of the three planes A, B, C did not employ the full symmetry of the figure. Any other "diagonal" set of 3 planes centred at points, one on each line A, B, C, one on each line D, E, F, would have done as well. Thus there are six equivalent projective conditions, shown by six sets of three concurrent lines in Figure 3.

These projective conditions have an interesting connection with plane conic curves. Delete any "diagonal" set of three vertices from the grid. The remaining six vertices form a hexagon, the edges of which are along the six lines of the grid. Each projective condition noted above is simply the statement that the three lines joining opposite vertices of one of these hexagons are concurrent. By Brianchon's theorem, this is equivalent to saying that the six lines are tangent to a plane conic. We will come back to a dual problem, and Pascal's theorem, when we discuss the mechanics of a complete bipartite structure, in section 4.2, below.

Which of these three methods is more suitable for a general algorithmic approach to the problem of obtaining a geometric description of special positions in which a plane figure lifts to higher dimensions? The first approach, the direct construction of the spatial object, followed by a general plane projection, seems like a good candidate, and will provide easy geometric solutions to simple problems. It is not immediately useful for a polyhedron unless we know how to construct it by slicing a tetrahedron. The second approach assumes rather that we know the solution in advance, or that we have ready access to enormously many theorems of projective geometry. The third approach, involving a search for higher-order syzygies, holds some promise as a general technique. A general programming strategy has, however, yet to be established.

4. Mechanics of linkages, rigidity of structures

A plane bar-and-joint framework $G = G(V, E)$ consists of a set $V = \{a, \ldots\}$ of $v = |V|$ *nodes* in positions $a = (a_1, a_2, 1), \ldots$ in the projective plane, together with a set E of $e = |E|$ *bars*, which are pairs of distinct nodes. An *infinitesimal motion* of a framework G is an assignment of free vectors $v_a, \ldots$ to the nodes, such that for every edge ab,

$$(v_a - v_b) \cdot (a - b) = 0.$$

We define the *rigidity matrix* of G as the $e \times 2v$ matrix with a row r_{ab} for each edge ab, as follows:

$$\begin{array}{c} \\ \mathbf{ab} \\ \vdots \\ \mathbf{bd} \end{array} \begin{array}{c} \begin{array}{ccccccc} \quad\mathbf{a} & & \qquad\qquad\mathbf{b} & & \ldots & & \quad\mathbf{d} \end{array} \\ \begin{pmatrix} a_1 - b_1 & a_2 - b_2 & b_1 - a_1 & b_2 - a_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & b_1 - d_1 & b_2 - d_2 & \ldots & d_1 - b_1 & d_2 - b_2 \end{pmatrix} \end{array}$$

An infinitesimal motion is representable as a $2v$-vector that is orthogonal to all rows of the matrix $M(G)$. The rows themselves may be regarded as particular assignments of vectors to the nodes, namely: equilibrium compression loads applied to individual bars.

For any e-vector ω with scalar entries ω_{ab}, the matrix product ωM is an equilibrium load which can be supported by the framework, and ω is a possible *response* of the framework to that load (a system of tensions and compressions which balances the external load at every vertex). The row space of $\mathcal{M}(G)$ is the space of equilibrium loads supported by the framework. The kernel $\{\omega; \omega M = 0\}$ of the linear transformation "right multiplication by $\mathcal{M}$" is the space of self-stresses in G, that is: systems of tensions and compressions in the bars that are in equilibrium with the zero load at every vertex.

A plane framework with at least two distinct vertices has a space of infinitesimal motions of dimension at least equal to 3. Every isometry (translation or rotation) of the plane induces an infinitesimal motion to the framework. If there are no other infinitesimal motions, we say the framework is *infinitesimally rigid.* A framework is *statically rigid* if and only if every equilibrium load on its vertex set is supportable by the framework. Infinitesimal and statical rigidity are equivalent notions.

A framework is *independent* if and only if it has no non-zero self-stress. A framework is *isostatic* if and only if it is independent and infinitesimally rigid. A framework is *generically isostatic* if and only if it is isostatic in some (and therefore in almost any) position in the plane. Plane frameworks are generically isostatic if and only if they have $2v - 3$ bars on v nodes, and no more than $2v' - 3$ bars on any subset of v' of those nodes. A good algorithm exists for determining whether a framework is generically rigid in the plane, but appropriate extensions of such methods to structures in 3-space have not yet been found [Crapo, 1989b].

4.1. PURE CONDITIONS

Generically isostatic frameworks are of particular interest from an invariant-theoretic point of view, because of the special positions in which they become dependent, and acquire infinitesimal mobility. We provide an example, to indicate the passage from the rigidity matrix to its associated *pure condition* for dependency of the framework. Let G be a framework consisting of two triangles ace, bdf, linked by three additional bars, ab, cd, ef. Its rigidity matrix is shown in figure 4.

$$\begin{array}{cc}
 & \begin{array}{cccccccccccc} a_1 & a_2 & b_1 & b_2 & c_1 & c_2 & d_1 & d_2 & e_1 & e_2 & f_1 & f_2 \end{array} \\
\begin{array}{c} \mathrm{ab} \\ \mathrm{ac} \\ \mathrm{ae} \\ \mathrm{bd} \\ \mathrm{bf} \\ \mathrm{cd} \\ \mathrm{ce} \\ \mathrm{df} \\ \mathrm{ef} \end{array} &
\begin{pmatrix}
a_1-b_1 & a_2-b_2 & b_1-a_1 & b_2-a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_1-c_1 & a_2-c_2 & 0 & 0 & c_1-a_1 & c_2-a_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_1-e_1 & a_2-e_2 & 0 & 0 & 0 & 0 & 0 & 0 & e_1-a_1 & e_2-a_2 & 0 & 0 \\
0 & 0 & b_1-d_1 & b_2-d_2 & 0 & 0 & d_1-b_1 & d_2-b_2 & 0 & 0 & 0 & 0 \\
0 & 0 & b_1-f_1 & b_2-f_2 & 0 & 0 & 0 & 0 & 0 & 0 & f_1-b_1 & f_2-b_2 \\
0 & 0 & 0 & 0 & c_1-d_1 & c_2-d_2 & d_1-c_1 & d_2-c_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_1-e_1 & c_2-e_2 & 0 & 0 & e_1-c_1 & e_2-c_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_1-f_1 & d_2-f_2 & 0 & 0 & f_1-d_1 & f_2-d_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_1-f_1 & e_2-f_2 & f_1-e_1 & f_2-e_2
\end{pmatrix}
\end{array}$$

Figure 4. The rigidity matrix $\mathcal{N}$ of the "triangular prism" graph.

For a model of the generic structure, we may take the values $a_1, a_2, \ldots, f_1, f_2$ to be independent transcendentals. The row space of the matrix $\mathcal{N}$ is the orthogonal complement of the row space of the matrix $\mathcal{M}$, whose rows give the motion vectors of three independent isometries of the plane, two translations and a rotation about the origin:

$$\begin{array}{c} \\ \mathbf{t_x} \\ \mathbf{t_y} \\ \mathbf{r} \end{array} \begin{array}{c} \begin{array}{cccccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{e} & \mathbf{f} \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ -a_2 & a_1 & -b_2 & b_1 & -c_2 & c_1 & -d_2 & d_1 & -e_2 & e_1 & -f_2 & f_1 \end{pmatrix} \end{array}$$

The Grassmann coordinates of the row spaces of $\mathcal{M}$ and $\mathcal{N}$ are Hodge star complements, so every 9×9 minor of the matrix $\mathcal{N}$ is, up to an overall scalar multiple λ, equal to a 3×3 minor (in the complementary set of columns) of the matrix $\mathcal{M}$, multiplied by ± 1 according to the parity of the split of the set of columns into two complementary parts. Say we wish to evaluate the $a_1 a_2 \ldots e_1$ minor of $\mathcal{N}$. By the above argument, we know it is equal to λ times the minor in the last three columns of $\mathcal{M}$. This minor is equal to $e_1 - f_1$. Looking at the first nine columns of the matrix $\mathcal{N}$, we see that there is indeed a factor of $e_1 - f_1$ in its determinant, and that the scalar λ is equal to the determinant of the 8×8 matrix $\mathcal{N}'$ obtained by deleting the row ef and columns e_1, e_2, f_1, f_2. This scalar λ, which depends of course on the positions of the points $a, \ldots, f$, is a projective invariant, called the pure condition for dependence of the framework [White and Whiteley, 1983, 1987]. It is clear that non-zero products occur in the expansion of the determinant λ only for "diagonals" of the 8×8 matrix $\mathcal{N}'$ obtained by matching all the edges (except ef) to components (indexed $_1$ or $_2$) of incident nodes (other than e and f). Each such diagonal occurs with a number of companion diagonals, obtained by interchanging the matched vertices and components for the edges matched to the two components of any node (if $ab \to b_1, bd \to b_2$ is part of a diagonal, it can be exchanged for $ab \to b_2, bd \to b_1$). These diagonals are of opposite sign, so the corresponding terms have the minor

$$\begin{vmatrix} b_1 - a_1 & b_2 - a_2 \\ b_1 - d_1 & b_2 - d_2 \end{vmatrix} = \begin{vmatrix} b_1 & b_2 & 1 \\ a_1 & a_2 & 1 \\ d_1 & d_2 & 1 \end{vmatrix} = [bad]$$

We see the polynomial λ in its familiar form, as a sum of products of brackets of triples of points. Think of each edge as *oriented* toward the node to which it is matched. The sign of each product of brackets is the parity of the set of *crossed* pairings $xy \to y, st \to s$, where $x \leq s < y < t$ in some standard order on the points. In the example of the stucture under consideration, there are two terms, corresponding to the matchings

$$\begin{array}{cccccccc} ab \to b & ac \to a & ae \to a & bd \to d & bf \to b & ce \to c & cd \to c & df \to d \\ ab \to a & ac \to c & ae \to a & bd \to b & bf \to b & ce \to c & cd \to d & df \to d \end{array}$$

The first matching has sign -1 because $ab \to b$ crosses $ac \to a, ae \to a$, while $bd \to d$ crosses $bf \to b, cd \to d, ce \to c$. The second matching has sign $+1$ because $ac \to c$ crosses $ae \to a, bd \to b, bf \to b$, while $cd \to d$ crosses $ce \to c$. We have

$$\begin{aligned} \lambda &= -[ace][baf][cde][dbf] + [abe][bdf][cae][dcf] \\ &= [ace][bdf]([abe][fcd] - [abf][ecd]) \\ &= [ace][bdf](ab \vee ef \vee cd). \end{aligned}$$

The degree of every node in the pure condition of a framework is equal to the number of incident edges, less 1.

The connection with scene analysis is made via Maxwell's theorem: the 1-skeleton of a spherical polyhedron (a planar 3-connected graph), represented as a bar-and-joint framework with no collinear faces, is dependent as a framework if and only if it is the projection of a proper spatial polyhedron (not entirely confined to a single plane). The pure condition for the framework is not equal to the polynomial invariant associated with the polyhedral projection. In the the case at hand, the latter polynomial would be simply $ab \vee ef \vee cd$. The extra bracket factors arise from the triangular faces ace, bdf. If one of these three-element sets of points, say ace, is collinear on a line L, the framework is dependent. The corresponding phenomenon in scene analysis is that a spatial realization exists in which the six points are coplanar on some plane P, and the three points ace are *also* coplanar on a distinct plane Q whose intersection with P is a straight line which projects down to the line L. We have called such figures "realizations of type II". It is in this context that Maxwell's theorem shows the relation between scene analysis and plane mechanics [Crapo and Whiteley, 1990b].

Attempts to generalize the Maxwell theorem to structures in 3-space, say in terms of projections of 4-polytopes, have not so far been successful. This question, together with the need for an algorithm to decide generic rigidity in 3-space, are the principal unsolved problems in structural mechanics.

4.2. BIPARTITE FRAMEWORKS

The most thoroughly understood class of frameworks (in any dimension) are the complete bipartite frameworks $K_{A,B}$. Their special positions occur when the vertex sets A or B are both dependent as sets of projective points, or when the set $C = (\langle A \rangle \cap B) \cup (\langle B \rangle \cap A)$ lies on unexpectedly many quadratic surfaces [Bolker and Roth, 1980]. Here $\langle A \rangle$ means the projective subspace spanned by the set A.

Take a close look at the complete bipartite graph $K_{3,3} = K_{ace,bdf}$ as a bar-and-joint framework in the plane [Whiteley, 1984]. It will be dependent if and only if its six vertices lie on a conic. (This includes the special case in which ace and bdf are both collinear triples). The polynomial of its pure condition

$$[def][abe][bcf][cda] - [dea][abe][bcf][cdf] + [dfa][abe][bce][cdf] - [efa][abd][bce][cdf]$$

factors as

$$(ab \wedge de) \vee (bc \wedge ef) \vee (cd \wedge fa),$$

an expression which occurs in the statement of Pascal's theorem: six points lie on a plane conic if and only if opposite sides of the hexagon $(abcdef)$ meet in collinear points. But there is something bizarre going on here. The condition "$abcdef$ lie on a conic" is symmetric in the six points, whereas the condition "opposite sides of the hexagon $(abcdef)$ meet at collinear points" depends on a choice of a "necklace" order of the six points. Indeed, there are 45 derived points of the form $ab \wedge cd, \dots cd \wedge ef$, and 60 necklace orders of the six points. For a general figure of six points on a conic, these 60 necklace orders determine 60 *distinct* lines, each containing exactly 3 of the 45 derived points. See Figure 5. If one of the six points moves off the conic determined by the other five, all 60 collineations will simultaneously fail! This example shows another aspect of the Cayley factorization problem. A seemingly simple invariant polynomial form, such as the above sum of four products of four brackets each, can have as many as 60 combinatorially and geometrically distinct Cayley factorizations! Any algorithm for non-linear Cayley factorization will have to select one of these 60 forms.

The condition that six points lie on a conic in the plane is equivalent to the condition that *the second symmetric powers of those six points are dependent.* The

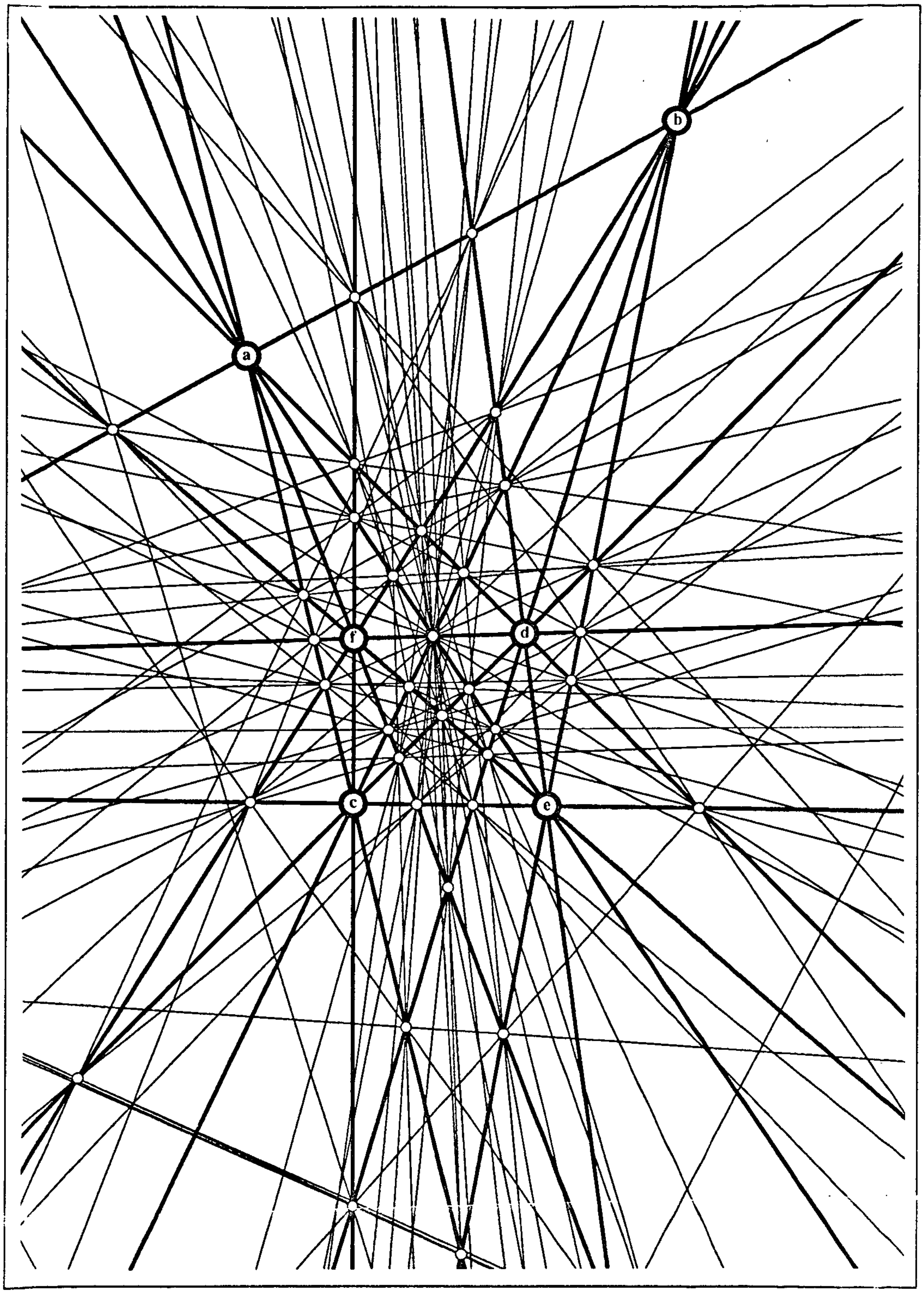

Figure 5. What Pascal's theorem really says!

second symmetric power $a^{(2)}$ of a point $a = (a_1, a_2, a_3)$ in the projective plane is coordinatized by a vector in rank 6,

$$a^{(2)} = (\, a_1^2 \quad a_1a_2 \quad a_2^2 \quad a_1a_3 \quad a_2a_3 \quad a_3^2 \,).$$

Consider the matrix $\mathcal{M}$, the columns of which are the symmetric powers of six points $a, \ldots, f$.

$$\begin{array}{c} \\ \mathbf{x^2} \\ \mathbf{xy} \\ \mathbf{y^2} \\ \mathbf{xz} \\ \mathbf{yz} \\ \mathbf{z^2} \end{array}
\begin{array}{c} \begin{array}{cccccc} \quad\mathbf{a} & \quad\mathbf{b} & \quad\mathbf{c} & \quad\mathbf{d} & \quad\mathbf{e} & \quad\mathbf{f} \end{array} \\
\begin{pmatrix}
a_1^2 & b_1^2 & c_1^2 & d_1^2 & e_1^2 & f_1^2 \\
a_1a_2 & b_1b_2 & c_1c_2 & d_1d_2 & e_1e_2 & f_1f_2 \\
a_2^2 & b_2^2 & c_2^2 & d_2^2 & e_2^2 & f_2^2 \\
a_1a_3 & b_1b_3 & c_1c_3 & d_1d_3 & e_1e_3 & f_1f_3 \\
a_2a_3 & b_2b_3 & c_2c_3 & d_2d_3 & e_2e_3 & f_2f_3 \\
a_3^2 & b_3^2 & c_3^2 & d_3^2 & e_3^2 & f_3^2
\end{pmatrix} \end{array}$$

The symmetric powers are dependent if and only if there is a dependence among the columns of $\mathcal{M}$, if and only if $\mathcal{M}$ is singular, if and only if there is a dependence among the rows of $\mathcal{M}$, that is, if and only if there is a homogeneous quadratic polynomial

$$Q(x, y, z) = q_{11}x^2 + q_{12}xy + q_{22}y^2 + q_{13}xz + q_{23}yz + q_{33}z^2$$

which is zero when (x, y, z) coordinatizes any one of the six points $a, \ldots, f$.

A final question concerning the invariant property that six points lie on a plane conic: what can be said about the 15 generic first-order syzygies among those 6 points? The set of 15 syzygies is of rank 3, so they are coplanar on some plane P in projective 5-space, rank 6. The five syzygies among any five points are of rank 2, and are therefore collinear on lines

$$L_a = \delta_{bcde}, \delta_{bcdf}, \delta_{bcef}, \delta_{bdef}, \delta_{cdef}$$

$$\vdots$$

$$L_f = \delta_{abcd}, \delta_{abce}, \delta_{abde}, \delta_{acde}, \delta_{bcde}$$

There are in general no other collineations among the 15 first-order syzygies. The syzygies $abcd, abef, cdef$ could, for instance, be collinear, but that would mean that the lines ab, cd, ef are concurrent. The only other thing which can happen is that two syzygies, say *abcd* and *abce* coincide. This means that *abc* is a collinear triple of points.

It seems reasonable to conjecture that *some symmetric statement concerning these* 15 *syzygies is equivalent to the fact that the six points lie on a conic.* If so, it will have to be a third-order syzygy. What is it?

4.3. CENTRES OF RELATIVE MOTION

The centres of relative motion of an infinitesimally flexible plane framework lie at the vertices of geometric configuration obtained by intersecting n hyperplanes in projective $(n-2)$–dimensional space, then projecting that figure into the plane. This observation provides a support for an algorithmic approach to the determination of all possible infinitesimal motions of a given framework. In the example in Figure 6, which is formed by selecting certain edges of a plane triangular grid, the parts labelled A and B are constrained to move together, as are the parts labelled C and D. Call those enlarged parts A and C, respectively. Then the centres of relative motion lie at positions

χ_{AC} at z,the point at infinity along the direction of their common bars,
χ_{AE} at s, the top vertex of component E,
χ_{AF} at y, the point at infinity along the direction of their common bars,
χ_{CE} at q, the top vertex of component D,
χ_{CF} at x, the point at infinity along the direction of their common bars,
χ_{EF} at r, the lower-left vertex of component E.

One may verify that these six points do indeed form a complete quadrilateral. The given framework is actually generically isostatic. The motion is possible because the framework is in special position. Such examples have been more thoroughly discussed in [Bideau et al, 1988]. Additional material on all matters discussed in this paper will be available in a forthcoming book [Crapo and Whiteley, 1990b].

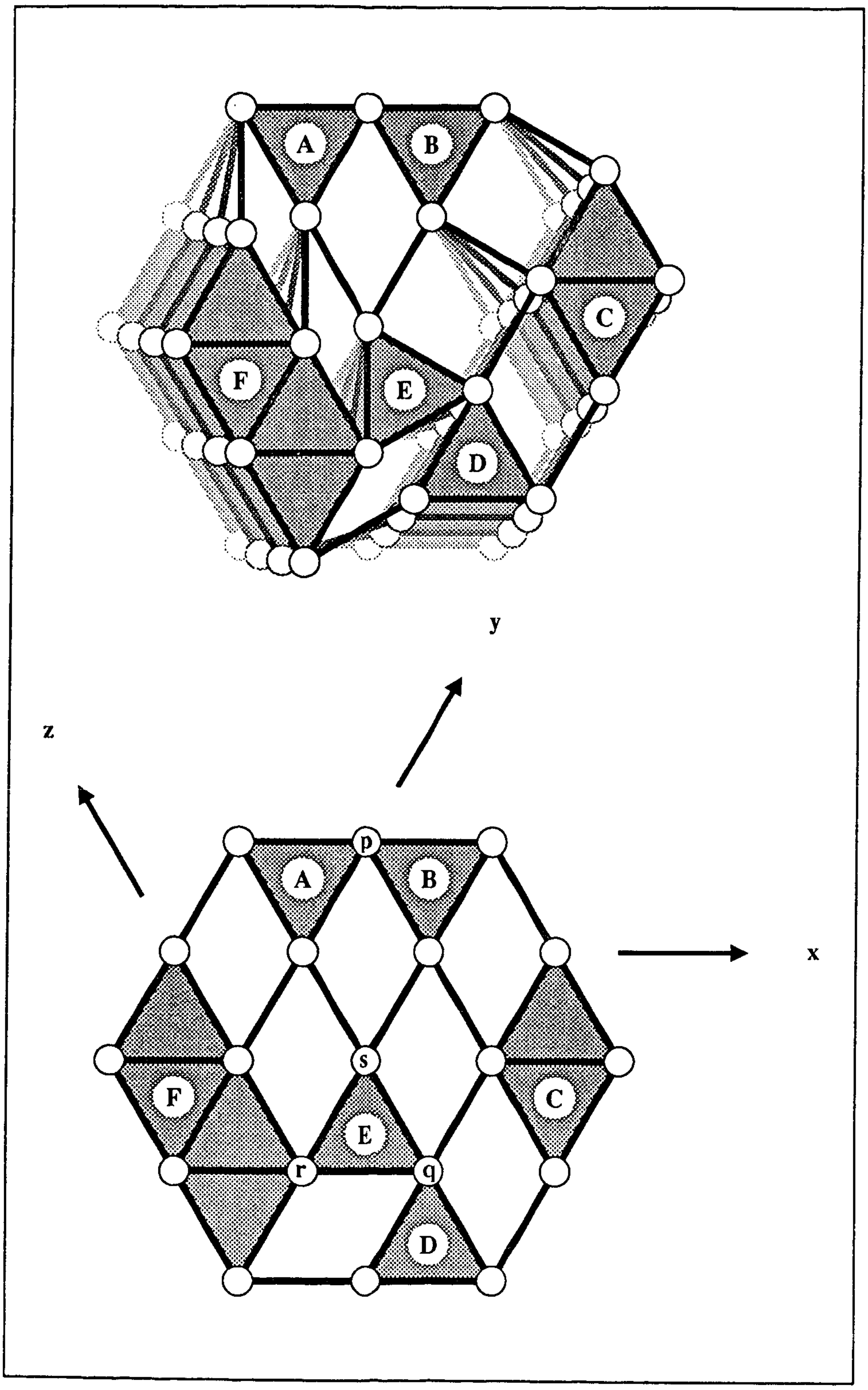

Figure 6. A possible finite motion, in special position, of a generically isostatic plane framework.

5. Bibliography

Janos Baracs, 1973-present. "Topologie Structurale", cahiers de cours, Université de Montréal.

Daniel Bideau, Henry Crapo, Etienne Guyon, Alex Hansen, Stéphane Roux, and Jean-Paul Troadec, 1988. "Non-local and Non-linear Problems in Mechanics of Disordered Systems: Applications to Granular Media and Rigidity Problems", Comptes rendus of the congress "Mechanics of Disordered Media", Sitges, Spain.

Ethan Bolker and Ben Roth, 1980. "When is a bipartite graph a rigid framework?", Pacific Journal of Math., 90, 27-43.

Henry Crapo, 1985. "The Combinatorial Theory of Structures: Lectures on the application of combinatorial geometry in architecture and structural engineering", in "Matroid Theory", Colloq. Math. Soc. János Bolyai 40, North-Holland, Amsterdam - New York, 107-213.

Henry Crapo and Juliette Ryan, 1986. "Spatial Realizations of Linear Scenes", Structural Topology 13, 33-68.

Henry Crapo, 1989a. "Applications de l'Homologie Géométrique", Comptes rendus des Journées Géométrie et Robotique, Toulouse, May 1988, Springer-Verlag, Lecture Notes in Computer Science.

Henry Crapo, 1989b. "On the Generic Rigidity of Plane Frameworks", submitted for publication.

Henry Crapo and Walter Whiteley, 1990a. "Plane Stresses and Projected Polyhedra", Structural Topology, to appear.

Henry Crapo and Walter Whiteley, 1990b. "The Geometry of Rigid Structures", Cambridge University Press, Encyclopedia of Mathematics and its Applications, to appear.

Peter Doubilet, Gian-Carlo Rota and Joel Stein, 1974. "On the Foundations of Combinatorial Theory IX: combinatorial methods in invariant theory", Studies in Applied Mathematics 53, 185-216.

Kokichi Sugihara, 1983. "A Unifying Approach to descriptive geometry and mechanisms", Disc. Appl. Math. 5, 313-328.

Tiong-Seng Tay and Walter Whiteley, 1985. "Generating Isostatic Frameworks", Structural Topology 11, 21-69.

Neil White and Walter Whiteley, 1983. "The Algebraic Geometry of Stresses in Frameworks", SIAM Journal of Algebraic and Discrete Methods 4, 481-511.

Neil White and Walter Whiteley, 1987. "The Algebraic Geometry of Motions of Bar-and-Body Frameworks", SIAM Journal of Algebraic and Discrete Methods 8, 1-32.

Walter Whiteley, 1984. "Infinitesimal Motions of a Bipartite Framework", Pacific Journal of Math. 110, 233-255.

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